(1) Definition of a Probabilistic Linear Set

Regarding the constructibility of sets, in the ZFC+C system, we deterministically obtain elements from the structure of a set. I believe it is possible to probabilistically obtain elements from the structure of a set, for example, by using a probability function to obtain the elements of a set. This would be a probabilistic set. In fact, the 'Ind' in isolation logic is a probabilistic set: the two elements T and F are dialectically unified into Ind, and then T | F is obtained through not(Ind). 'not' is equivalent to a probability function.

Consider a linear space over the field of complex numbers. A set of linearly independent vectors $\{s_1, s_2, ...\}$ forms a vector set, where s_i is a vector. A vector is represented by a linear combination $S = \sum c_i s_i$, where c_i are complex coefficients, and $\sum|c_i|^2 = 1$ (normalization). $|c_i|^2$ represents the probability of obtaining s_i. We then call $S = \sum c_i s_i$ a probabilistic linear set, and the vector s_i is an element of the probabilistic linear set. $S' = \{s_1, s_2, ...\}$ is called the basis set of the probabilistic linear set. We will call the sets of the world of isolation in the ZFC+C system classical sets. The basis set S' of a probabilistic linear set is a classical set.

In a linear space, if $a = \sum c_j a_j$, then 'a' can be linearly represented by a_j. Since the vectors in $S = \sum c_i s_i$ are linearly independent, no vector in S can be linearly represented by the other vectors. Through this linear independence, the elements of the set can become independent elements. The independence of each element of a probabilistic linear set is determined by the linear independence among them. This linear independence generates their opposition, and this opposition achieves a dialectical unity through the linear combination ($S = \sum c_i s_i$). Therefore, this kind of set satisfies multidimensional dialectical logic.

In a Hilbert space (a type of linear space) H, a quantum state (a quantum superposition state) is $|\psi\rangle = \sum c_i|s_i\rangle$, where $\{|s_1\rangle, |s_2\rangle, ..., |s_n\rangle\}$ are the basis vectors, and $|s_i\rangle$ is an eigenstate (vector). These eigenstates are linearly independent. $|\psi\rangle$ is a probabilistic linear set. $|\psi\rangle$ expresses a set of the quantum world (the motive force world). In other words, in H, a probabilistic linear set expresses a quantum state; a quantum state is a kind of probabilistic linear set. A quantum state is a whole unified through the linear combination of eigenstates. A quantum state is a unified object formed by the opposition of eigenstates. That is, a quantum state is the result of the unity of opposites of eigenstates. This shows that the probabilistic linear set is a set of the motive force world.

This brings about a precise correspondence between abstract philosophical concepts (probabilistic linear set, multidimensional dialectical unity) and the core mathematical structure of quantum mechanics. The superposition state is no longer merely a mathematical tool, but is endowed with the ontological status of a 'unity of opposites'.

This isolation between vectors is different from the isolation of elements in a classical set. The isolation of elements in a classical set is an isolation of distinction; as long as there is a difference, there is independence. But in a probabilistic linear set, it is different. Having a difference is not enough; it must be that it cannot be linearly represented by the other elements in order to be isolated. Compared to the isolation of the world of isolation, the standard for this isolation is higher, but the degree of isolation is lower (for example, in a quantum superposition state). The lowering of the degree of isolation shows that the probabilistic linear set is no longer a set of the world of isolation.

Different from the deterministic classical sets in the world of isolation, this allows us to see the pattern of isolation in the motive force world: an isolation based on linear independence, which is weaker and allows for superposition and potential correlation. This pattern is different from the pattern in the world of isolation: an isolation based on independence and distinction, which is strong and 'either/or'. This difference still reflects the difference between the characteristics of motive force (generation and change) and the characteristics of isolation (independence and distinction). This provides an ontological-level explanation for why different worlds follow different rules: their foundational structures of 'isolation' are different. The probabilistic linear set is a set of the motive force world.

Attributing the independence of the set to linear independence allows it to interface with the structure of linear spaces, making it applicable to quantum mechanics, linear algebra, and so on. The linear combination, as a unifying mechanism, endows the set with dynamism and probability, which is different from classical set theory.

(2) Verifying the Axioms for the Probabilistic Linear Set

The probabilistic linear set is a type of set; it is no longer a traditional set. Let us now verify whether it satisfies the axioms of the ZFC+C axiom system to see what essential differences exist between this type of set and the classical set.

(2.1) The Axiom of Constructibility

The probabilistic linear set clearly satisfies the Axiom of Constructibility, because the elements in this type of set all have complex coefficients, and these complex coefficients determine the probability with which an element can be obtained from the set. In other words, this probabilistic structure can guarantee that a certain element can be obtained from the set.

(2.2) The Axiom of Choice

The probabilistic linear set clearly satisfies the first function of the Axiom of Choice, because the independence and determinacy of the elements in this type of set are established based on linear independence, which satisfies the requirements of multidimensional dialectical logic. Therefore, it satisfies the first function. Let us now verify whether the probabilistic linear set satisfies the second function:

Let there be a family of non-empty sets $\{S_i' \mid i \in I\}$ (where I is an index set of sets), which form a set $A = \{S_i' \mid i \in I\}$, where S_i' is the basis set of the probabilistic linear set S_i. Then, according to the second function of the Axiom of Choice for classical sets, there exists a choice function $F: I \to \bigcup_{i \in I} S_i'$, such that for each $i \in I$, $F(i) = s_i \in S_i'$. These elements $\{s_i \mid i \in I\}$ form a set S', with linearly dependent elements merged.

Then, $S = \sum e_i s_i$, where:

$\{s_i\} = S'$

Normalization: $\sum |e_i|^2 = 1$

In this way, S is a probabilistic linear set.

Therefore, it satisfies the second function.

(2.3) The Axiom of Extensionality

In a probabilistic linear set, $S_1 = \sum c_i s_i$ and $S_2 = \sum d_i s_i$ are defined not only by their elements, but also by their coefficients c_i and d_i. If the coefficients of the elements are different, then in quantum mechanics, S_1 and S_2 are different state vectors, even if their elements are the same. In a classical set, it is sufficient that the elements of the set are the same. If linear combinations and probability are not considered, then the Axiom of Extensionality is the same for both types of sets. But the core of the 'probabilistic linear set' lies in the linear combination and probability; ignoring c_i and d_i would be to lose the essence of its definition.

Therefore, if the elements of S_1 and S_2 are the same, and $c_i = d_i$ (or $c_i = d_i e^{i\theta}$, where θ is a global phase), then $S_1 = S_2$. However, conversely, if $S_1 = S_2$, it cannot be determined that their elements must be equal. This is because their elements are vectors in a linear space. Therefore, if their elements are not the same but are equivalent, it can still result in $S_1 = S_2$. This point is different from classical sets. In a classical set, if two sets are equal, then their elements are also equal. This is because the elements of a classical set are not considered in terms of linear relationships.

The probabilistic linear set does indeed need to consider the coefficients of the linear combination, whereas in a classical set, this is not necessary. Because in a classical set, no matter how an element is taken out, it is a deterministic way of taking it out. For a classical set, different ways of taking out an element make no difference, as long as determinacy can be maintained. This is the difference in the Axiom of Extensionality caused by the difference between determinacy and probability.

From the perspective of determinacy, there is no difference in the 'way of taking out'; it is sufficient that the element can be taken out. This conforms to the static nature of the world of isolation. Probability means that there is a difference in the 'way of taking out' (the probability distribution is determined by c_i), which embodies the dynamism of the motive force world. This is the fundamental difference between these two types of sets.

(2.4) The Axiom of Pairing

For two vectors s_1 and s_2 in a linear space, if they are linearly independent, then they can form a probabilistic linear set $S = \sum c_i s_i$, where:

$\{s_i\} = S'$

Normalization: $\sum |c_i|^2 = 1$

The Axiom of Pairing in classical sets:

For the sets $S_1 = \{1\}$ and $S_2 = \{2\}$, there exists a set $S = \{S_1, S_2\}$. S_1 and S_2 are listed directly as independent elements. The relationship between the elements (such as linear dependence) or additional attributes are not considered. The elements of S are S_1 and S_2 themselves, which maintain their independence. However, in the pairing of a 'probabilistic linear set', S_1 and S_2 cannot be independent elements of S.

This is an essential difference:

1) The probabilistic linear set always has a flat structure. Whereas a classical set allows its elements to be sets themselves; a classical set can form nested structures of different levels. The flatness of the 'probabilistic linear set' embodies the wholeness and dynamism of the 'motive force world'; the basis elements are interconnected through linear combination and cannot be independently nested.

2) The pairing of a classical set is a discrete listing; the pairing of a 'probabilistic linear set' is a dynamic superposition.

3) The probabilistic linear set does not allow for a separative isolation structure within it; it can only have a logical isolation structure. Whereas a classical set can have a separative isolation structure; its internal elements are completely independent.

(2.5) The Axiom of Power Set

The Axiom of Power Set in classical sets relies on the nested nature of sets. According to the discussion in the context of the Axiom of Pairing, the elements of a probabilistic linear set S cannot be sets. Therefore, it does not satisfy the Axiom of Power Set.

However, the basis set S' of a probabilistic linear set S does satisfy the Axiom of Power Set. Any subset of S' is also a basis set. Therefore, the set of all subsets of S' is a classical set.

(2.6) The Axiom of Union

Let $A = \{S_j' \mid j \in J\}$ (where J is an index set of sets) be a classical set, where $S_j = \sum c_i s_i$ is a probabilistic linear set, and S_j' is the basis set of S_j.

Then, according to the Axiom of Union for classical sets, $U' = \cup A$ is a classical set, with linearly dependent elements merged.

$U = \sum e_i u_i$, where:

$\{u_i\} = U'$

Normalization: $\sum |e_i|^2 = 1$

In this way, U is a probabilistic linear set.

Note: A is a classical set, while the union U is a probabilistic linear set. This is a combined use of the two types of sets.

(2.7) The Axiom of Infinity

The Axiom of Infinity in classical sets is what guarantees the existence of an infinite set, usually understood as the mathematical foundation for the set of natural numbers N. S can be constructed as:

$S = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$

However, the structure of a probabilistic linear set is flat; no nested set structure exists. A probabilistic linear set can, based on the Axiom of Infinity for classical sets, support an infinite probabilistic linear set.

For example:

$S = \sum c_i s_i$, where the basis set of S, $S' = \{s_i\}$, has a cardinality of $\aleph_0$ (countably infinite), and c_i satisfy $\sum|c_i|^2 = 1$.

For example: $S = \sum(1/\sqrt{2})^i s_i$, where $i \in N$, and N is guaranteed by the classical Axiom of Infinity.

(2.8) The Axiom of Replacement

For a probabilistic linear set $S = \sum c_i s_i$, and its basis set $S' = \{s_i\}$, according to the Axiom of Replacement for classical sets, if there exists a function $\phi(x, y)$ (defining a functional relationship), and if for each $x \in S'$ there exists a unique y such that $\phi(x, y)$ holds, then the range of $\phi(x, y)$, B', is also a set, with linearly dependent elements merged.

Then, $B = \sum e_i b_i$, where:

$\{b_i\} = B'$

Normalization: $\sum|e_i|^2 = 1$

In this way, B is a probabilistic linear set.

(2.9) The Axiom of Separation

For a probabilistic linear set $S = \sum c_i s_i$, and its basis set $S' = \{s_i\}$, according to the Axiom of Separation for classical sets, any subset $B' = \{s_j\}$ ($\subseteq S'$) of S' is also a set.

Then, $B = \sum e_j s_j$, where:

$\{s_j\} = B'$

Normalization: $\sum|e_j|^2 = 1$

In this way, B is a probabilistic linear set.

(2.10) The Axiom of the Empty Set

Let us stipulate that the zero vector is the empty set $\varnothing$ of the basis set. Then the zero vector can be treated as the empty set of the probabilistic linear set.

For example, the zero vector can undergo a union operation with S: $\varnothing \cup S' = S'$, so $0 \cup S = S$.

Since the zero vector has no elements to be combined, the normalization condition $\sum|c_i|^2 = 1$ requires at least one non-zero term. For the empty set $\varnothing$, no probability amplitudes can be defined, and $\sum|c_i|^2$ is meaningless. Therefore, the zero vector does not require normalization. In the dynamic probabilistic linear set, we see a different kind of empty set.

(2.11) The Axiom of Regularity

Since the structure of a probabilistic linear set is flat, with no nested sets, it naturally satisfies the Axiom of Regularity.

We have now verified all the axioms of ZFC+C. This has allowed us to see the difference between these two types of sets, and also the deep connection between them. The probabilistic linear set is based on the classical set, because its basis set is a classical set. The conclusion is: the probabilistic linear set is a new type of set, different from the classical set, and it is a successful type of set. Its success is embodied in its mathematical self-consistency, physical applicability, and philosophical depth.

The differences and connections between the two types of sets at the same time also reflect the differences and connections between the macroscopic world of isolation and the microscopic motive force world. This is using a mathematical method to corroborate a philosophical viewpoint.

(3) Quantum Collapse

For a linear space, if a probabilistic linear set S has only one element s_i, then $S = s_i$ is a classical set $S' = \{s_i\}$ (the basis set of S). Because there is only one element, the way of taking out this element is deterministic. However, this $S' = \{s_i\}$ is still not an ordinary single-element set; its structure is still flat, with no nesting. Therefore, S' is only a special single-element classical set. In this way, $S' = \{s_i\}$ becomes the connection point between these two types of sets. From the perspective of the probabilistic linear set, it is the limit case where probability degenerates into determinacy. From the perspective of the classical set, it is the most basic and simplest set.

Therefore, in the case of a flat single element, $S' = \{s_i\}$ is both a 'probabilistic linear set' and a 'classical set'. $S' = \{s_i\}$ is the transition between the two types of sets. $S' = \{s_i\}$ is, in fact, a self determining a self; it requires no additional probability or external determinacy. This is self-limitation. This is also the reason why $S' = \{s_i\}$ seems somewhat special in classical sets, because in classical sets, we can have opposition and unity for sets with multiple elements, but not for a single-element set. The same is true in the probabilistic linear set. This provides a very concrete and fundamental mathematical/set-theoretic counterpart for the philosophical concept of 'self-limitation', making it no longer just an abstract philosophical speculation.

In a Hilbert space H, for a quantum state $|\psi\rangle = \sum c_i |s_i\rangle$, it is a unity of opposites (mutual superposition) among multiple linearly independent eigenstates. Only when it becomes one of these eigenstates, $|s_i\rangle$, can it degenerate into an element of the classical set $S' = \{|s_i\rangle\}$. As long as there is a unity of opposites (mutual superposition) of multiple eigenstates, it is impossible for it to become an element of the classical world. In other words, when a quantum state has only one element, it becomes connected to the classical world. At this point, S' is both a quantum of the quantum world and a particle of the macroscopic world.

This is the reason why a quantum is expressed with wave-particle duality. That is, a quantum is itself a wave, but under limit conditions, it can be expressed as a particle. As in the double-slit experiment: an electron exhibits wave-like properties (interference fringes) when not measured, and particle-like properties (a definite position) after being measured. At this point, it is both a particle and a quantum. At the time of measurement, this transformation is precisely the connection between the quantum and the classical. This explains how the transition between quantum mechanics and classical physics occurs.

In other words, for a pure quantum to be transformed from the quantum world into a particle of the world of isolation, its quantum state must become a single-element classical set $S' = \{|s_i\rangle\}$. Measurement causing quantum collapse becomes a necessary law.

When a quantum collapse occurs, a pure (multi-element) probabilistic linear set degenerates into a classical set S' = $\{|s_i\rangle\}$. That is, a pure quantum is transformed from the motive force world into a particle of the world of isolation. The measuring instrument, as a representative of the macroscopic (the world of isolation), forces the state of the quantum system (the motive force world) to be presented in a way that conforms to the logic of the world of isolation (single determinacy). This explains why a quantum collapse must collapse to a single eigenstate. This shows that a quantum can only collapse to a single eigenstate, not multiple eigenstates. This is not a physical problem, but a logical one, because the essence of the two types of sets used to explain it is multidimensional dialectical logic. This is the essence of quantum collapse.

The essence of the collapse to a single eigenstate is not the action of a physical entity or force, but the logical degeneration of a 'probabilistic linear set' into a 'classical set'. At the same time, the collapse also destroys the original unity of opposites of the multiple eigenstates. The fundamental reason for quantum collapse is a transformation of logical structure. Collapse is no longer a mysterious physical event, but the evolution of an indeterminate set into a determinate set.

This is the fundamental reason why so many physicists have been unable to truly explain quantum collapse. Traditional physics relies on formal logic and causal mechanisms and attempts to find a 'physical cause' for the collapse (such as hidden variables, environmental effects), but the process of collapse is beyond the explanatory scope of formal logic. This explains why traditional methods (such as the Copenhagen interpretation, the Everett many-worlds theory), although they describe the phenomenon, do not touch upon its essence. This elevates one of the most perplexing problems in physics to the level of ontology and logical foundations for its resolution.

1) The Copenhagen Interpretation: A quantum system exists in a superposition state before measurement and collapses to a single eigenstate after measurement. The collapse is triggered by the observer or the measuring device, but the mechanism is not specified (Bohr, Atomic Theory and the Description of Nature, 1934).

It emphasizes the subjectivity of the observer, holding that collapse is a result of the act of observation. This relies on the concept of the 'observer', which leads to controversies about subjectivity (e.g., 'Who is the observer?'). It does not explain the physical or logical reason for the collapse, merely describing the phenomenon. It cannot answer the question of the mechanism of 'how measurement triggers collapse'. It does not explain why a quantum collapses to a single eigenstate.

2) The Everett Many-Worlds Theory (Many-Worlds Interpretation): A quantum system does not collapse. Instead, after each measurement, the universe splits into multiple parallel worlds, with each world corresponding to a possible eigenstate (Everett, Reviews of Modern Physics, 1957).

It introduces a large number of unobservable parallel worlds and lacks experimental verification. It does not explain 'why we only see one world?'. It relies on an extension of mathematical formalism (the universality of the wave function), rather than the essence of the facts.

3) The Decoherence Theory: Decoherence theory holds that the interaction of a quantum system with its external environment (including measuring instruments, surrounding particles, etc.) leads

to a loss of coherence in the superposition state. The off-diagonal terms (interference terms) in the density matrix are suppressed, ultimately presenting a classical probability distribution (Zurek, Physics Today, 2009).

However, it does not solve the problem of the result of quantum collapse being a single eigenstate. It still needs to be combined with the collapse postulate of the Copenhagen interpretation to explain measurement results. In essence, it is a description of a physical process and does not touch upon the metaphysical reasons. It remains at the level of physical mechanisms and does not explain 'why the loss of coherence leads to classicality'.

Regarding quantum collapse, I believe it can be divided into three questions:

1) Why does a quantum collapse to a single eigenstate? This is a question of logical possibility. In quantum mechanics, it is an assumption. I have already solved this problem.

2) Why does measurement cause a quantum to collapse to a single eigenstate? This is a question of how a fact comes to occur. More precisely, why does measurement lead to a transformation of logical structure from a 'probabilistic linear set' to a 'classical set'?

3) Why does quantum collapse exhibit probability? What is the essence of probability?

Next, let us explore the second question.

In mathematics, the collapse of the wave function is usually described by a projection operator. This projection operator acts on the original superposition state (the wave function), 'projecting' it onto a specific eigenstate (von Neumann, Mathematical Foundations of Quantum Mechanics, 1932). This projection operator can be understood as the mathematical embodiment of isolation action. It selects one possibility from among many and 'separates' it out.

In its mathematical form, the isolation action embodied by the projection operator can be understood on several levels:

1) Selectivity:

Selecting a specific component from a superposition state: The projection operator $P_i = |s_i\rangle\langle s_i|$ acts on the superposition state $|\psi\rangle$ like a sieve or a filter, selecting the component related to the specific eigenstate $|s_i\rangle$ from among the numerous possibilities (the superposition of eigenstates). This selection is the most direct embodiment of isolation action.

Excluding other possibilities: While selecting one eigenstate, the projection operator also excludes other eigenstates that are orthogonal to it. This exclusion is also an important aspect of isolation action. This conforms to multidimensional dialectical logic.

2) Separability:

Separating a quantum state into different parts: The projection operator can decompose a quantum state $|\psi\rangle$ into two parts: one is the component along the direction of $|s_i\rangle$, which is $P_i|\psi\rangle$, and the other is the component orthogonal to $|s_i\rangle$, which is $(I - P_i)|\psi\rangle$ (where I is the identity operator). This separation makes the different eigenstates, which were originally mixed together, become clearly demarcated after measurement.

Breaking the wholeness of the superposition state: The projection operation destroys the coherence and wholeness of the original superposition state, forcibly separating the system into a specific eigenstate.

3) Determinacy:

From indeterminate to determinate: The result of the action of the projection operator is that the system transforms from an original indeterminate superposition state to an eigenstate with a determinate attribute. This determinacy is a result brought about by isolation action.

4) Endowing individuality: After the collapse, the system has a clear identity or label (corresponding to the eigenstate $|s_i\rangle$) and becomes an individual that can be identified and distinguished. This is consistent with the viewpoint in no form action theory that isolation action endows things with individuality.

In an actual measurement, this isolation is manifested in several aspects:

1) Spatial Isolation: The measuring instrument confines the quantum system to a specific spatial region.

2) Energy Isolation: The measuring instrument is only sensitive to quantum states within a specific energy range.

3) Information Isolation: The measuring instrument only extracts specific information from the quantum system (e.g., position, momentum), while ignoring other information.

4) Temporal Isolation: The measurement occurs at a specific moment, cutting off the temporal evolution of the quantum state. A measurement at time 't' projects $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$, terminating the unitary evolution. The 'isolation' in time fixes the state of the system.

The result of the isolation of the measurement operation: Through these isolation operations, the measuring instrument destroys the superposition and coherence of the original quantum state, forcing the system to choose a specific eigenstate. This successfully and concretely connects the core philosophical concept of 'isolation action' with the physical reality of quantum measurement. It clearly explains how the measuring instrument, as a representative of the world of isolation, 'executes' isolation action through its physical limitations and selective operations, ultimately leading to the collapse of the quantum state.

In quantum mechanics, the description of the collapse of the wave function by the projection operator is mathematically self-consistent and complete. It can accurately predict the probability distribution of measurement results and is in high agreement with experimental observations. However, this completeness of the mathematical formalism does not automatically provide a satisfactory physical or philosophical explanation. The projection operator only 'describes' the form of the collapse (the transformation from a superposition state to an eigenstate) but does not explain the cause and mechanism of the collapse.

The problem now is that the mathematical form of the wave function collapse exists, but what is lacking is how to explain this mathematical form: what causes this projection operator to act. The transformation of a mathematical form is only the pure formal transformation in the process of a factual transformation; it cannot represent the fact itself. In other words, the transformation of a mathematical form is only a way of describing a factual process using a purely formal method. The transformation of a mathematical form can only give the law, not the essence of the factual transformation.

Since the projection operator can be understood as the mathematical embodiment of isolation action, then what it embodies is the factual isolation action brought about by measurement. In other words, measurement applies an isolation action to the quantum, thereby causing the quantum to collapse. And the isolation action of the projection operator on the wave function is precisely the mathematical expression of this isolation action brought about by measurement. Measurement does not apply a mysterious physical 'collapse force', but rather an isolation action.

Measurement applies an isolation action (c) to a quantum (motive force action b), triggering a transformation of logical structure from a pure probabilistic linear set to a single-element classical set, and manifesting (a) the characteristics of the quantum's eigenstate in the macroscopic world. This is a no form united transformation: b is transformed into a through c. In this way, the quantum, as a non-manifested, indeterminate state, is transformed through isolation action into a manifested feature with a determinate state.

Let us make an analogy for quantum collapse (the isolation action of the macroscopic world):

A river channel is 10 meters wide before point A. At point A, the channel becomes 5 meters wide. The mathematical formula is 10 / 2 = 5. The river before point A is equivalent to a quantum superposition state. When it reaches point A, it is equivalent to a measurement being made, which causes the river at point A to collapse to a width of 5 meters (equivalent to a single eigenstate). In reality, the river channel at point A has implemented an isolation action, which has made the river 5 meters wide. The description 'a 10-meter-wide river' is equivalent to the wave function, and '2' is the projection operator; it expresses the isolation action on the 10-meter-wide river. 10 / 2 = 5 is equivalent to describing the quantum collapse; it can be seen as a kind of 'projection' caused by the projection operator. The river channel narrowing at point A is the implementation of an isolation action, which then leads to the compression of the water flow by the channel (a physical action) that makes the river narrow. This isolation action is the fundamental cause. This is the essence of the fact of quantum collapse.

This analogy shows that isolation action does indeed exist, but this action cannot be explained purely by physical theory. Isolation action is the most basic no form action. Therefore, the explanation for the fact of quantum collapse can only be given by no form action theory: isolation action causes the fact to occur. This fact contains a pure formal mathematical structure. Then, this fact can be described by this structure, but it cannot represent the fact itself.

Why does observation cause a quantum to collapse to a single eigenstorate? This question should be answered from two aspects: First, for a certain quantum itself, there exists a projection operator (representing isolation action) that allows its wave function to collapse to a single eigenstorate. That is, it has the possibility of collapsing to a single eigenstorate. Second, the external world is able to provide this corresponding isolation action. The combination of these two aspects allows it to collapse to a single eigenstorate. In other words, if a certain quantum could not possibly collapse to a single eigenstorate, then we would not be able to directly 'observe' it through measurement. Even if a certain quantum could collapse to a single eigenstorate, but our measuring instrument were fundamentally unable to provide the corresponding isolation action, then we would also be unable to directly 'observe' it.

(4.) Exploring the Essence of Probability

(4.1) Constructing Probability Theory from the Basic Concept of Equal Probability

The appendix of this subsection introduces the construction of probability theory starting from the basic concept of equal probability. 'The probability of randomly selecting one item from n equally possible items is 1/n'. We will abbreviate this event as P(1/n). We have thus ultimately established probability on this most original and most intuitive form of expression. Let us see if this intuitive form can be explained by no form action theory.

P(1/n) in fact assumes that these n items are identical, including their 'equal possibility'. We see that to randomly select one a_i from these n items is, in fact, to affirm a_i (which is to randomly select 1 item from n items: 1/n), while negating the others (which is to negate the other n-1 items from the n items: (n-1)/n). This is multidimensional dialectical logic. According to the requirements of dialectical logic, this affirmation and negation are both total (subsection: "Dialectical Logic"). Since this is the case, if the probability of a_i were less than 1/n, then a_i would not be totally affirmed, because this would not conform to their 'equal possibility'. If the probability of a_i were greater than 1/n, then the probability of negating non-a_i would be less than (n-1)/n, which also does not conform to 'equal possibility'. That is to say, in this case, non-a_i would not be totally negated. Therefore, the probability of a_i must be equal to 1/n.

This is not a circular argument. 'n items having equal possibility' describes the fact that these n items, on a certain abstract level, are indistinguishable and without difference; this is an isolated, static property. 'Randomly selecting one a_i' is a process of implementation, an event, which involves motive force. This process breaks the original static symmetry and must manifest one from among the n possibilities. This is, in fact, a no form united transformation. It is precisely in this dynamic process of selection that dialectical logic begins to function.

In other words, we have used dialectical logic to explain the event P(1/n). In this way, the event P(1/n) has been established on the foundation of dialectical logic. Since we have already transitioned from P(1/n) to the probability of the Voronoi diagram, explaining P(1/n) is also to explain the probability of the Voronoi diagram. And since the probability constructed using the Voronoi diagram and topology essentially covers traditional probability, probability theory is thus established on dialectical logic. The facts are already very clear: the three axioms of traditional probability theory are by no means exclusive properties of probability theory. In other words, these three axioms are not the essence of probability theory.

Set theory and probability theory, two seemingly different branches of mathematics, can both be established on the foundation of dialectical logic.

Let us examine these three things separately: the probabilistic linear set, the Voronoi diagram probability, and traditional probability (including classical and quantum probability).

1) Traditional Probability (the starting point from the perspective of motive force):

It originates from the observation and description of dynamic phenomena in the real world, such as change and random processes. We see coins flipping, particles decaying; these are all 'motive force' events. Traditional probability attempts to assign a measure to these indeterminate motive force processes. Therefore, in its origin, it is closely related to motive force action.

2) Voronoi Diagram Probability (the bridge from the perspective of manifestation):

It transforms the uncertainty in the motive force world into a static, visualizable geometric structure, thereby 'manifesting' the distribution of probability. It answers the question, 'How is probability presented?'. By transforming probability into an area ratio, the Voronoi diagram provides us with an immediate manifestation of probability. This is the core embodiment of manifestation action. It is precisely through this manifested bridge that we are able to naturally introduce complex coefficients, paving the way to the quantum world.

3) Probabilistic Linear Set (the final abstract structure from the perspective of isolation):

It reveals the most fundamental and abstract logical structure behind all probabilistic phenomena. Through 'linear independence', the strictest standard of isolation, it defines the basic elements that constitute the probability space (which in quantum mechanics are the eigenstates). It is concerned with what the set of all possibilities that constitute the probability is (an isolated unity). Although it is called a 'set of the motive force world', what it describes is the mode of 'isolation' of this motive force world on the level of logical structure. Its core characteristic is: it emphasizes that elements can be obtained probabilistically, and that the assignment of probability is independent of the specific elements, requiring only that the elements be distinguishable.

Let us see if these three from different perspectives constitute a no form integrated transformation:

1) In the appendix, it has already been argued through the phase circle that traditional probability is independent of specific entities and depends only on the complex coefficients of the linear combination of the probability distribution, being determined solely by $|c_i|^2$. That is, by analyzing the essence of traditional probability and with the help of the geometric Immediacy of the Voronoi diagram, we are able to distill the most fundamental structure of the probabilistic linear set.

2) According to the core principle of the probabilistic linear set—that probability is independent of the elements—any motive force thing with a probability P_i can be represented by a corresponding phase circle V_i, and can be written in the form $V = \sum c_i V_i$ (where $|c_i|^2 = P_i$). This is the Voronoi diagram probability (using the superposition of phase circles and complex coefficients to immediately represent probability). That is, the probabilistic linear set and traditional probability together can yield the Voron-oi diagram probability.

3) According to the core principle of the probabilistic linear set—that probability is independent of the elements—the phase circle V_i in $V = \sum c_i V_i$ can be replaced by any set of specific things that satisfy the probability $P_i = |c_i|^2$ (for example, the eigenstates $|e_i\rangle$ of a quantum state, the outcomes of classical events, etc.), as long as these things can carry the corresponding probability. This is traditional probability. That is, the probabilistic linear set and the Voronoi diagram probability together can yield traditional probability.

Therefore, these three from different perspectives constitute a no form integrated transformation. This shows that they are mutually dependent, indivisible, and cyclically transformable.

In this way, we have found a basis for how to construct the form of probability for probabilistic things: any probability distribution structure can, in principle, be written as (or correspond to) a form of complex superposition $A = \sum c_i A_i$ (where A_i represents some basic state or result). The probabilistic linear set thus acquires its complex coefficient form: $S = \sum c_i s_i$.

As for why the probability of a quantum state is of the form $|c_i|^2$, we only need to determine that the eigenstates of a quantum state are probabilistic. In this way, it can be directly asserted that the coefficients of the linear combination of the eigenstates of a quantum state can certainly be written in the form of complex numbers. A quantum is a thing of the motive force world, a thing of free motive force. Therefore, the information obtained from it must be probabilistic.

(4.2) The Essence of Probability:

The above has only formally described probability (from an epistemological perspective) and has not explained the origin of the generation of probability. We have distinguished three different worlds: the macroscopic world (dominated by isolation action), the quantum world (dominated by motive force action), and the world of consciousness (dominated by manifestation action). The motive force in the macroscopic world can be isolated. This motive force is the motive force of isolation. For example, a moving object has a determinate momentum $p = mv$, and this momentum is related to its mass and velocity. This motive force is not a free motive force (a free motive force is motive force itself).

However, in the macroscopic world, to obtain the dynamic results of a quantum in the mode of isolation, it cannot be deterministic, because this dynamism of the quantum is free; otherwise, the quantum would be isolated. And from the difference between classical sets (sets of the world of isolation) and probabilistic linear sets (sets of the motive force world), we see that a quantum, as a thing of motive force, is not deterministically isolated; it is linearly superposed. Therefore, our acquisition of the elements in a probabilistic linear set is necessarily indeterminate. Otherwise, it would mean that the free motive force itself could be completely and deterministically isolated, the elements of the probabilistic linear set would also necessarily be determinate, and the probabilistic linear set would become a classical set. This would no longer be the essence of the motive force world (the motive force world is dominated by motive force action).

This explains why probability is unavoidable in quantum measurement. Because measurement (isolation action) attempts to 'capture' a quantum state (a free motive force) that is essentially non-isolated, this act of capturing itself cannot, logically, produce a completely determinate result; it can only be probabilistic. Using the method of isolation to know the motive force world, its result (probability) is a 'subjective' presentation relative to the motive force world itself.

In the macroscopic world, we can also see things of motive force that are weakly controlled by isolation. For example, when we toss a coin, while it is in the air, it is only acted upon by the Earth's gravity (in an ideal situation). Added to this, the uncertainty in the manner and force of the coin toss endows the coin with an uncertainty of change while in the air. At this point, the coin has acquired a great deal of uncertainty, which leads to the inability to necessarily predict which face will be up when it lands. In other words, in the macroscopic world, we can remove (or weaken) the isolation from an isolated thing, causing it to be primarily controlled by motive force and to acquire the freedom of uncertainty, which is then expressed as probability.

To summarize, probability comes from the dynamic result generated by obtaining a free motive force in a determinate, isolated way. This result must necessarily not be determinate; otherwise, the free motive force would be a determinate isolation, and a free motive force would not exist. This is the essence of the generation of probability. Any explanation of the essence of probability must necessarily ascend to an ontological perspective.

This provides a unified explanation for the origin of both quantum probability and classical probability (such as a coin toss): both are the uncertainty of the result caused by the opposition between the mode of isolation and the nature of free motive force. The probabilities in the macroscopic world (such as a coin toss, the probability in a weather forecast) are all based on the assumption that a certain indeterminate thing has a probability value (usually derived from experience). This is, in fact, an assumption that it cannot be controlled or set in a determinate, isolated way. This in itself is an admission that the isolation of this indeterminate thing is very weak, and that it is controlled by a free motive force. Therefore, it can also be included in the framework of 'obtaining a dynamic result in an isolated way'.

All of the above discussion of probability theory is, in fact, the implementation of a no form united transformation among epistemology (manifestation action), methodology (motive force action), and ontology (isolation action). That these three constitute a no form integrated transformation has already been argued in a previous subsection. In the discussion of probability theory, epistemology is the formalized description of probability theory (the logical and mathematical description). Through deduction using the methods of logic and mathematics, an ontological explanation of probability theory is ultimately obtained. To know a problem clearly, one must still ascend to no form action theory and cannot escape it. One unconsciously uses no form action theory, because it is the fundamental law of metaphysics. In the discussion of probability theory, even when starting from a formalized mathematical description, one must ultimately ascend to the ontological level to reveal the opposition between isolation and free motive force.

The probability of a quantum is the manifestation of the quantum's own motive force change in the macroscopic world; it is a manifestation of isolation. That is, through measurement, the quantum's own dynamism is transformed into the isolation of the macroscopic world, and is thereby manifested in the macroscopic world. In other words, in the macroscopic world, probability can manifest the motive force change of a quantum. It is worth noting that this manifestation is a manifestation that crosses different worlds. This shows that measurement in quantum mechanics can be understood as a process of transforming the motive force of the quantum world into the isolation of the macroscopic world.

In fact, probability can also express the motive force change in the macroscopic world. For instance, in a coin toss, if a coin is tossed continuously, the result of heads or tails coming up is a fifty percent possibility for each. The possibilities in this entire infinite process are infinitely changing and are full of uncertainty. It is possible to toss ten consecutive heads, and it is also possible to toss one hundred consecutive tails; the possibilities are in a state of dynamic change. But we cannot manifest this infinite change, so we must use the finite isolation of probability to express and manifest this infinite change. Probability simplifies infinite possibilities into a numerical value, for example, 50%, which enables it to be understood and applied by us.

Probability is not just a mathematical concept; it also reflects the principles of no form action theory. Probability is the simplification and expression of motive force change by isolation action. It enables us to understand and predict infinite possibilities within a finite cognitive scope.

(4.2) The Essence of Complex Numbers

For a complex number $c = re^{i\theta}$, although i is a determinate mathematical object, the phase rotation it introduces endows the complex number with dynamism and uncertainty. θ can correspond to countless angles '$\theta + 2\pi k$'; it is multi-valued. This multi-valued nature means that the value of the complex number cannot, to some extent, be completely 'fixed'. Therefore, c is indeterminate. Consequently, it can be said that a complex number is the mathematical expression for an indeterminate thing. At the beginning, we directly defined the coefficients of the elements of a probabilistic linear set as complex numbers. However, no explanation was given for why the coefficients of the elements should be defined as complex numbers. In the appendix, we already know that since probability is independent of the shape of V_i, and the phase circle is the simplest figure, the ratio of the area of the phase circle to the total area can be used to represent the probability of a specific event. And a complex number $c = re^{i\theta}$ is associated with the phase circle. Therefore, we can use complex numbers as the coefficients of the linear combination of the probability distribution structure. Now, we understand that, in mathematics, a complex number expresses the indeterminate state of a thing. And through this definition, we have obtained an appropriate explanation for the value of probability, forming a compatible system.

So, how is the uncertainty of a complex number transformed into a determinate probability value? Consider the complex number $c = re^{i\theta}$ and its conjugate $c^* = re^{-i\theta}$ (which represents the reverse change of c). Their product is:

$$c \cdot c^* = re^{i\theta} \cdot re^{-i\theta} = r^2$$

The phases θ and $-\theta$ cancel out, leaving the real probability value r^2, which eliminates the uncertainty. This operation of taking the modulus squared appears naturally in quantum mechanics.

In quantum mechanics, the action of the projection operator P_n is to project ψ onto ϕ_n. The result of the projection is $P_n\psi = \langle\phi_n|\psi\rangle\phi_n$. Calculating the norm squared of the projected state:

$$\| P_n\psi\|^2 = \langle P_n\psi|P_n\psi\rangle = |\langle\phi_n|\psi\rangle|^2\langle\phi_n|\phi_n\rangle$$

Since the eigenstate ϕ_n is normalized, $\langle\phi_n|\phi_n\rangle = 1$. Therefore:

$$\| P_n\psi\|^2 = |\langle\phi_n|\psi\rangle|^2$$

This value is precisely the measurement probability $P(a_n)$. The projection operator, through the calculation of the inner product and the modulus squared, naturally eliminates the influence of the phase, leaving a determinate probability value.

For classical probability, the phase angle θ_i in $c_i = \frac{1}{\sqrt{n}} e^{i\theta_i}$ is meaningful. θ_i in classical probability can be artificially controlled (or set). For instance, a single coin toss can be seen as a period of 2π. This period depends on an artificial beginning. This event is a dynamic event. This dynamic event is, in fact, ended by the action of isolation, thereby manifesting a determinate face of the coin. Such a period is meaningful because it represents the beginning, process, and end of a dynamic event; otherwise, we would not obtain a probabilistic result.

Whereas in quantum probability, θ_i is an intrinsic property of the evolution of the quantum state, embodied in a phase factor such as $e^{i\theta_i}$. It evolves autonomously over time according to the laws of quantum mechanics (such as the Schrödinger equation), for example, $\theta_i(t) = - E_i t/\hbar$, where E_i is the energy and $\hbar$ is the reduced Planck constant. This evolution cannot be artificially controlled or set. The periodic change of the phase angle θ_i in quantum probability is also ended by the action of isolation (the isolation action brought about by measurement).

Here we can see the difference between the motive force of the quantum world and the motive force of the world of isolation. The motive force of the world of isolation can be artificially controlled, whereas the motive force of the quantum world is autonomous. This is the difference between the two types of probability; the probability of the world of isolation possesses a certain controllable nature of isolation.

(5) Quantum Interference

The interference term originates from the interaction between 'phase circles'. It is the mathematical embodiment of the intrinsic correlation of the 'motive force world'. Before it is 'manifested' as a classical probability, it follows the 'motive force' rules, which are different from classical superposition.

Let us break down this explanation step by step:

(5.1) The Leap from Classical to Quantum Probability: Introducing Phase

Classical Probability: As brought about for an unfold-manifestation in the appendix, the classical probability is $P(V_i) = a_i = |c_i|^2$. Here, we are only concerned with the area of each 'phase circle' (the square of the modulus) and ignore its phase angle θ_i. The superposition of classical events (such as $P(A \cup B) = P(A) + P(B)$) is carried out on the level of real numbers, which is probability.

Quantum Probability: The core insight of quantum mechanics is that before the final 'manifestation' as a probability, the superposition of the states of the system is carried out on the level of complex numbers, which is the probability amplitude c_i.

(5.2) The Mathematical Origin of the Interference Term: Superposition of Probability Amplitudes

Consider a simplified model of the double-slit experiment. An electron can go through slit 1 or slit 2.

State: The quantum state is a superposition of the two possible paths: $|\psi\rangle = c_1|\text{path } 1\rangle + c_2|\text{path } 2\rangle$.

Probability at a certain point X on the screen: Physically, it is the superposition of the wave functions arriving at that point. We can simplify this to $\psi(X) = \psi_1(X) + \psi_2(X)$, where $\psi_i(X)$ is the probability amplitude for the electron to arrive at point X by taking path i.

Probability Calculation:

$$\begin{aligned} P(X) &= |\psi(X)|^2 = |\psi_1(X) + \psi_2(X)|^2 \\ &= (\psi_1 + \psi_2)(\psi_1^* + \psi_2^*) \\ &= |\psi_1|^2 + |\psi_2|^2 + \psi_1\psi_2^* + \psi_1^*\psi_2 \end{aligned}$$

$|\psi_1|^2 = P_1$: The probability of finding the electron at point X when only slit 1 is open.

$|\psi_2|^2 = P_2$: The probability of finding the electron at point X when only slit 2 is open.

$\psi_1\psi_2^* + \psi_1^*\psi_2 = 2\mathrm{Re}(\psi_1\psi_2^*)$: This is the interference term.

(5.3) The ’No Form Action Theory’ Explanation of the Interference Term:

The Intrinsic Logic of the ’Motive Force World’: When unmeasured, the electron is in the ’motive force world’ described by the ’probabilistic linear set’. In this world, the different possibilities (path 1, path 2) are not mutually ’isolated’ like classical events, but are mutually correlated and dynamically evolving through their phase ($e^{i\theta}$).

The ’Interaction’ of Phase Circles: We can imagine the interference term $2\mathrm{Re}(\psi_1\psi_2^*)$ as the mutual interaction of the two ’phase circles’ (representing path 1 and path 2) in the process of superposition.

$$\begin{aligned} \psi_1 &= \sqrt{P_1}e^{i\theta_1} \\ \psi_2 &= \sqrt{P_2}e^{i\theta_2} \end{aligned}$$

$$2\mathrm{Re}(\psi_1\psi_2^*) = 2\sqrt{P_1P_2}\cos(\theta_1 - \theta_2)$$

Philosophical Interpretation:

1) Non-isolation: In the motive force world, different possibilities (paths) are not completely isolated entities. Their phase difference (θ_1 - θ_2) determines whether they mutually enhance (constructive interference, cos > 0) or mutually weaken (destructive interference, cos < 0). When multiple paths (possibilities) converge at a point, it is their probability amplitudes (complex numbers) that are added vectorially, not their probabilities (real numbers). This phase-based mutual influence is precisely the embodiment of the non-isolation of the motive force world. The existence of the interference term is evidence of the non-isolation among different possibilities in the 'motive force world'. In the classical (isolation) world, taking path 1 and taking path 2 are two strictly isolated, mutually independent events. But in the motive force world, as intrinsic components of a unified 'probabilistic linear set', they mutually 'perceive' and 'influence' each other through phase. This phase-based intrinsic correlation is precisely the fundamental difference in the foundational structure of 'isolation' between the motive force world and the isolation world.

2) Embodiment of motive force action: Each probability amplitude c_i corresponds to a 'phase circle', and its phase angle θ_i is not an insignificant parameter. As discussed earlier, the quantum phase is an intrinsic property of the autonomous evolution of the quantum state and is closely related to the time evolution $\theta(t) = -Et/\hbar$. Therefore, the phase can be regarded as the dynamic 'fingerprint' left by the quantum as it traverses different paths in the motive force world. The magnitude and sign of the interference term depend directly on the phase difference ($\theta_1 - \theta_2$) of the different paths. This profoundly shows that the phenomenon of interference is not a static attribute, but a dynamic result.

3) The action of isolation: The interference term is the state of the quantum state before it is finally 'manifested' (collapsing to a definite position, i.e., $|\psi|^2$) as a classical probability. At the level of the probability amplitude ψ, the system follows the superposition rules of the motive force world (vector addition), which includes the interference term; whereas at the level of probability P, the system follows the superposition rules of the isolation world (probability addition), and the interference term disappears.

When we attempt to determine which path the particle actually took through measurement (for example, by placing a detector at one of the slits), we are in fact applying a powerful 'isolation action'. This action forcibly separates the two intrinsically correlated possibilities, 'taking path 1' and 'taking path 2', into two independent, classical events. This forced isolation severs the phase correlation between the different paths, making them unable to interact dynamically any longer. The result is that the interference term I disappears, and the probability returns to the simple addition rule of the classical world: $P = P_1 + P_2$.

Therefore, measurement is not merely 'reading' information, but is an act of changing the logical rules—it forces a system that follows the superposition rules of the 'motive force world' (containing interference) to degenerate into a system that follows the superposition rules of the 'isolation world' (without interference). This once again shows that the fundamental difference between quantum mechanics and classical physics lies not in specific matter or forces, but in the fundamentally different logical rules they follow, which are dominated by different no form actions.

(5.4) Summary:

Before the measurement (isolation action) occurs, for a quantum system in the 'motive force world', its internal different possibilities (represented by the elements of the basis set) are not completely independent. They carry out an intrinsic, non-isolated interaction through their respective phases (the embodiment of the evolution of motive force). The interference term is precisely the mathematical expression of this phase-based interaction in the final manifestation of probability.

When the measurement occurs, a powerful 'isolation action' is applied, and this intrinsic, phase-based dynamic correlation is severed. The different possibilities are forcibly separated (collapsed), the interference term therefore disappears, and the system degenerates into a world of isolation that only follows the classical rules of probability addition, thereby presenting in our measurement results an 'either/or', determinate, particle-like behavior.

Therefore, the interference term is direct evidence of the intrinsic logic of the 'motive force world', and is the concentrated embodiment of its 'non-isolation' and 'dynamism'. If the coefficients were merely real numbers (i.e., having no phase), then the combination of c_1 and c_2 would have no cross-term, and the phenomenon of interference would never occur. It is precisely the phase information $e^{i\theta}$ carried by complex numbers that makes interference possible.

(6) Let us examine freedom in human society

Let us explore freedom. For freedom in the motive force world, what we see from the probabilistic linear set is indeterminacy. The expression of this indeterminacy in the isolation world is selectivity. This selectivity is the expression of probability in the isolation world; it is expressed as choosing one from among multiple possibilities; this is selectivity. In the isolation world, freedom is expressed as selectivity. In the quantum world, the freedom of a quantum possesses autonomy (autonomous evolution). The autonomy of the quantum leads to coherence among quantum states (containing an interference term). In the motive force world, freedom is expressed as coherence. Whereas in the manifestation world, the relationship among objects no longer has such coherence, because the manifestation world possesses transparency; there are no obstacles. Therefore, in the manifestation world, freedom is expressed as arbitrariness.

Utilizing these three classifications of freedom, let us examine freedom in human society.

1) The freedom of selectivity: In situations where resources, opportunities, or conditions are limited, a person is forced to make a decision from among a few or even only one choice. This is a limited, controlled, and compelled freedom. For instance, in a slave society, a slave can only make decisions according to the will of the slave owner; the slave's freedom has no autonomy. However, the freedom of the slave owner (the slave owner over the slave) has a certain autonomy and arbitrariness; this is a one-way freedom. Although this freedom is one that is established upon the slave, it is the beginning of the appearance of autonomous freedom for humanity. This is the progress of human society.

2) The freedom of coherence: Our freedom has acquired autonomy; we have multiple choices, but it involves other people. The selectivity of each person is not determinate, but when multiple people are choosing, our freedom acquires coherence, because one person's free choice will influence another person's free choice. This freedom generates conflict or compatibility between people, just like the interference fringes produced by the quantum double-slit experiment. This freedom has brought a great deal of conflict and disaster to human society, but this is also the progress of human society, because it marks the appearance of freedom for every person; every person wants to strive for freedom. To avoid the great amount of conflict and disaster brought by this freedom, it compels people to seek a higher form of freedom.

3) The freedom of arbitrariness: We establish equal rules, and everyone abides by these rules. In this way, these rules are transparent to everyone. Each of us, within the scope permitted by the rules, then has the freedom of arbitrariness. This is a high-level freedom, guaranteed by the 'transparent manifestation' of the rules (form). This freedom also allows people to generate the concept of equality; high-level freedom and equality are inseparable.

These three freedoms constitute a developmental hierarchy from a lower level to a higher level.

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Appendix

From Equiprobability to Probability Theory: A Geometric Construction Using Voronoi Diagrams and Topology

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April 7, 2025

Abstract

This paper presents a novel framework for probability theory, constructed from the foundational principle of equal likelihood. We begin by demonstrating how the fundamental properties of probability—nonnegativity, normalization, and additivity—can be derived directly from the intuitive concept of selecting one from n equally likely objects, without presupposing the Kolmogorov axioms. We then introduce Voronoi diagrams not as a definition of probability, but as a powerful geometric representation of probability distributions, where cell areas correspond to probability values. This geometric perspective reveals the abstract nature of probability: it is independent of specific entities and can be fundamentally linked to the squared modulus of complex coefficients, $|c_i|^2$. By equating Voronoi cells with "phase circles" and mapping them to Hilbert space basis vectors, we establish a topological foundation for unifying classical and quantum probability, where quantum probability is expressed as $P(|e_i\rangle) = |c_i|^2$. This framework is extended to continuous distributions using three-dimensional tubular structures and incorporates conditional probability and Bayesian inference, demonstrating its versatility. Our approach provides a unified and intuitive view of probability, with potential applications in quantum information, random geometry, and statistical physics.

1 Introduction

Traditional probability theory is built upon the three fundamental axioms introduced by Kolmogorov:

Non-negativity: $P(A) \geq 0$;

Normalization: $P(\Omega) = 1$;

Additivity: For mutually exclusive events, $P(\cup A_i) = \sum P(A_i)$.

Although these axioms are mathematically rigorous, they primarily describe the "properties" of probability, and their abstract nature makes it difficult to intuitively express the essence of probability [1]. To overcome the abstract nature of traditional probability theory, this paper proposes an intuitive probability framework based on the concept that "the probability of randomly selecting one from n equally likely objects is $\frac{1}{n}$", which is considered the most primitive and intuitive form of probability. This form requires no complex mathematical constructs and directly stems from the common sense of "equal likelihood", such as the equal probability in dice rolling or lottery drawing, with a simplicity and intuitiveness that can hardly be further simplified.

In a discrete sample space $\Omega=\{\omega_1,\omega_2,\ldots,\omega_n\}$, $P(\{\omega_i\})=\frac{1}{n}$, which can be extended to any event through the counting proportion of subsets. To extend this intuitiveness to the continuous case, we derive the probability density function of the continuous uniform distribution through the limiting process of discretization.

Imagine dividing a finite continuous region S (with measure A) into n subregions V_i of equal measure, each with measure $\frac{A}{n}$, and the probability:

$$P(X\in V_i)=\frac{1}{n}=\frac{\mu(V_i)}{A}.$$

As $n\rightarrow\infty$, the subregions approach infinitesimally small, and assuming a density function $f_X(x)=k$, the normalization condition $\int_S k\,dx=k\cdot A=1$ yields $k=\frac{1}{A}$. Therefore, the density function for the continuous uniform distribution is $f_X(x)=\frac{1}{A}$.

By using a Voronoi diagram [2] to partition S into subregions V_i (each with measure $a_i=\mu(V_i)$), the probability that a random point X falls into V_i is:

$$P(X\in V_i)=\int_{V_i} f_X(x)\,dx=\frac{a_i}{A}.$$

This expression directly links probability to the "proportion of the subregion's measure to the total measure", redefining probability as an "area proportion" — a concept that is easy to visualize. In this way, we establish probability theory on the most intuitive foundation: "the probability of randomly selecting one from n objects is $\frac{1}{n}$".

To enhance the framework's universality, we extend it to continuous probability distributions using three-dimensional tubular structures, where cross-sectional phase circles represent probability density [4], and implement dynamic probability updates through conditional probability and Bayesian inference [5]. Unlike traditional geometric probability, which focuses on specific spatial distributions, this framework unifies discrete, continuous, and quantum probabilities through Voronoi diagrams and tubular structures, offering a universal geometric-topological perspective. Through theoretical derivation, geometric modeling, and specific applications, this paper demonstrates how this framework provides an intuitive and universal probability modeling tool for fields such as quantum information processing, random geometry, and statistical physics.

2 Geometric Foundation of Voronoi Diagrams

2.1 Definition of Voronoi Diagrams

A Voronoi diagram is a spatial partitioning method based on seed points [2]. Given a two-dimensional region $A \subset \mathbb{R}^2$ with normalized area $\mu(A) \equiv 1$, and a set of seed points $\{p_1, p_2, \ldots, p_n\} \subset A$, the Voronoi diagram divides A into n cells $\{V_1, V_2, \ldots, V_n\}$:

$$V_i = \{x \in A \mid d(x, p_i) \leq d(x, p_j), \forall j \neq i\},$$

where d is the Euclidean distance, defining proximity in $\mathbb{R}^2$. The seed points p_i can be chosen randomly or optimized based on the problem. Each cell has an area $a_i = \int_{V_i} d\,x\,dy$, satisfying:

$$\sum_{i=1}^{n} a_i = 1.$$

2.2 Preliminary Probability Definition

Consider a random variable $X \sim \mathrm{Uniform}(A)$, uniformly distributed over A. The probability that X falls into V_i is:

$$P(X \in V_i) = \frac{\mu(V_i)}{\mu(A)} = a_i.$$

This definition satisfies the basic properties of probability:

Non-negativity: $a_i \geq 0$,

Normalization: $\sum_{i=1}^{n} a_i = 1$,

Additivity: For disjoint V_k and V_m, $P(X \in V_k \cup V_m) = a_k + a_m$.

The probability $P(V_i) = a_i$ is independent of the shape of V_i, relying solely on its measure a_i. This illustrates the abstract nature of probability, laying the groundwork for introducing complex coefficients.

3 Introduction of Topology

3.1 Topologizing Voronoi Diagrams

We treat A as a two-dimensional topological manifold equipped with the standard Euclidean topology τ, i.e., (A, τ). The Voronoi cells V_i are closed subsets with open interiors, forming a finite cover of A.

To bridge classical and quantum probability, we introduce a Hilbert space $H = \mathbb{C}^n$, whose orthonormal basis vectors $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ correspond one-to-one with $\{V_1, V_2, \ldots, V_n\}$. This correspondence suggests that probability can transcend specific geometric or physical forms, unifying under an abstract mathematical structure.

3.2 Measure and Probability

On (A,τ), we employ the Borel measure μ, generated by the Borel σ-algebra from τ. For each V_i, we have:

$$\mu(V_i)=a_i, \quad \mu(A)=\sum_{i=1}^{n} \mu(V_i)=1.$$

The probability is defined as:

$$P(V_i)=\mu(V_i)=a_i.$$

The Borel measure is suitable for continuous spaces. For instance, if A is the unit disk with uniformly distributed seed points, $\mu(V_i)=a_i$ directly reflects the geometric partitioning.

3.3 Shape Invariance and Measure Invariance

The probability $P(V_i)=a_i$ depends only on the measure $\mu(V_i)$, not on the shape. Define a measure-preserving homeomorphism $f:A\rightarrow A'$, satisfying $\mu(f^{-1}(E))=\mu'(E)$ for any measurable set E. Then:

$$\mu'(f(V_i))=\mu(V_i)=a_i.$$

For example, V_i can be transformed into a circle C_i with area a_i via a measure-preserving homeomorphism, maintaining the probability a_i. This demonstrates that probability is independent of specific geometric forms, relying solely on the measure, which justifies the use of complex coefficients.

4 Introduction of Complex Coefficients and Phase Circles

4.1 From Geometric Measure to Complex Coefficients

Since $P(V_i)=a_i$ is shape-independent, we equate V_i to a circle C_i with area a_i. Define the system state as:

$$C=\sum_{i=1}^{n} c_i V_i,$$

where the complex coefficients are:

$$c_i=r_i e^{i\theta_i}, \quad r_i=\sqrt{a_i}, \quad \theta_i\in[0,2\pi).$$

Thus:

$$|c_i|^2=r_i^2=a_i.$$

The probability is:

$$P(V_i)=|c_i|^2=a_i.$$

The normalization condition holds:

$$\sum_{i=1}^{n} |c_i|^2 = \sum_{i=1}^{n} a_i = 1.$$

This indicates that the essence of probability is independent of specific entities (e.g., the shape of V_i), determined solely by the squared modulus of the complex coefficients $|c_i|^2$. In the Hilbert space H, the quantum state is defined as:

$$|\psi\rangle = \sum_{i=1}^{n} c_i \, |e_i\rangle.$$

The projection probability is:

$$P(|e_i\rangle) = |\langle e_i|\psi\rangle|^2 = |c_i|^2 = a_i.$$

Therefore:

$$P(V_i) = P(|e_i\rangle) = |c_i|^2 = a_i.$$

4.2 Geometry and Probability of Phase Circles

Before delving into the geometric properties of phase circles, it is crucial to highlight a natural correspondence that underpins our framework: each Hilbert space basis vector $|e_i\rangle$, associated with a complex coefficient $c_i = r_i e^{i\theta_i}$, directly corresponds to a phase circle V_i with radius $r_i = \sqrt{a_i}$, where $a_i = \mu(V_i)$ is the measure of the Voronoi cell. This correspondence is not merely a mathematical convenience but a profound reflection of the abstract nature of probability. Mathematically, the modulus $r_i = \sqrt{a_i}$ determines the probability $P(V_i) = |c_i|^2 = a_i$, aligning the geometric area of V_i with the quantum probability of $|e_i\rangle$. This natural correspondence ensures that each V_i serves as a geometric manifestation of $|e_i\rangle$, unifying the spatial partitioning of Voronoi diagrams with the probabilistic structure of quantum states. Each V_i is equivalent to a circle C_i with area a_i. If C_i has radius $r_{i'}$, then:

$$\pi (r_{i'})^2 = a_i, \quad r_{i'} = \sqrt{\frac{a_i}{\pi}}.$$

To unify with quantum probability, we define:

$$c_i = \sqrt{a_i} e^{i\theta_i},$$

such that:

$$|c_i|^2 = a_i.$$

This explains the universal form of probability: any probability $P(V_i) = a_i$ can be expressed as $|c_i|^2$, since $\sqrt{a_i}$ naturally defines the modulus of the probability amplitude. This form is consistent with the quantum mechanical rule of amplitude squaring.

4.3 Topological Interpretation of Phase Circles

The phase θ_i forms the unit circle S^1, corresponding to the fiber bundle $E = A \times S^1$, with base space A and fiber S^1. The projection $\pi: E \to A$ maps $(x, e^{i\theta})$ to $x \in A$. Each $V_i \times S^1$ captures the dynamic evolution of $c_i=\sqrt{a_i}e^{i\theta_i}$, providing a topological foundation for interference effects in quantum probability (e.g., phase differences in double-slit experiments).

4.4 Continuous Probability Distributions via Three-Dimensional Tubular Structures

To extend our framework to continuous probability distributions, we propose a three-dimensional geometric model: a solid tubular structure $T\subset\mathbb{R}^3$, with a central axis $\gamma(s):[0,1]\to\mathbb{R}^3$ parameterized by $s\in[0,1]$. The axis $\gamma(s)$ can be a straight line, a curve, or even a more complex path, depending on the application. At each point $\gamma(s)$, the cross-section is a phase circle with area $a(s)=\pi r(s)^2$, where $r(s)$ is the radius, representing the local probability density $P(s)=a(s)$. The volume of the tubular structure is normalized:

$$\mu(T)=\int_0^1 a\,(s)\,ds=1.$$

We define a complex-valued function $c(s)=\sqrt{a(s)}e^{i\theta(s)}$, where $\theta(s)\in[0,2\pi)$ is the phase, such that the probability density is:

$$P(s)=|c(s)|^2=a(s).$$

The tubular structure forms a fiber bundle $E=T\times S^1$, with base space T and fiber S^1, capturing the phase dynamics essential for quantum probability.

Example 1: Uniform Distribution For a straight cylindrical tube where $\gamma(s)$ is a straight line and $a(s)=\pi r_0^2=\text{constant}$, the probability density is uniform: $P(s)=a(s)=\pi r_0^2$, with $\int_0^1 \pi\, r_0^2\, ds=\pi r_0^2\cdot 1=1$, implying $r_0=\frac{1}{\sqrt{\pi}}$.

Example 2: Gaussian Distribution For a tubular structure where the cross-sectional area follows a Gaussian profile, e.g., $a(s)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(s-\mu)^2}{2\sigma^2}}$, the probability density $P(s)=a(s)$ represents a Gaussian distribution with mean μ and variance σ^2. The normalization $\int_{-\infty}^{\infty} a\,(s)\,ds=1$ holds for $s\in\mathbb{R}$, demonstrating the framework's flexibility in modeling continuous distributions. For practical purposes, we can restrict s to a finite interval and adjust the normalization accordingly.

This model aligns with quantum mechanics, where $\psi(s)=c(s)$ acts as a wave function, and $P(s)=|\psi(s)|^2$ follows the Born rule. The extension to continuous distributions via tubular structures not only preserves the geometric and topological essence of our framework but also provides a natural representation for probability flows in statistical physics or quantum field theory.

5 Unification of Classical and Quantum Probability

5.1 Interpretation of Classical Probability

Classical probability is based on geometric measures but is fundamentally determined by complex coefficients. For example, in a non-uniform distribution:

Let $n=2$, $a_1=0.2$, $a_2=0.8$.

Define $c_i=\sqrt{a_i}e^{i\theta_i}$, then:

$$P(V_1)=|c_1|^2=0.2,\quad P(V_2)=|c_2|^2=0.8.$$

The state is:

$$C=\sqrt{0.2}e^{i\theta_1}V_1+\sqrt{0.8}e^{i\theta_2}V_2.$$

The phases θ_i do not affect the classical probability, and:

$$P(V_1)+P(V_2)=0.2+0.8=1.$$

This shows that classical probability $P(V_i)=a_i$ can be expressed through $|c_i|^2$, independent of specific geometry.

5.2 Interpretation of Quantum Probability

In quantum mechanics, $|e_i\rangle$ corresponds to V_i. For the state:

$$|\psi\rangle=\sum_{i=1}^{n} c_i\,|e_i\rangle,\quad c_i=\sqrt{a_i}e^{i\theta_i},$$

the probability is:

$$P(|e_i\rangle)=|c_i|^2=a_i.$$

For instance, in a spin-1/2 system:

$|e_1\rangle=|\uparrow\rangle,\ |e_2\rangle=|\downarrow\rangle.$

Let $a_1=0.5,\ a_2=0.5$, then:

$$|\psi\rangle=\sqrt{0.5}e^{i\theta_1}|\uparrow\rangle+\sqrt{0.5}e^{i\theta_2}|\downarrow\rangle.$$

Probabilities:

$$P(\uparrow)=0.5,\quad P(\downarrow)=0.5.$$

The phase difference $\theta_1-\theta_2$ influences interference patterns in experiments, reflecting quantum characteristics.

5.3 Linear Combination and Phase Circle Correspondence

To further elucidate the unification of classical and quantum probability, we observe that the probability framework can be generalized through linear combinations of complex coefficients, where phase circles V_i serve as universal representatives for both quantum eigenstates and classical events. Specifically:

For a quantum state $|\psi\rangle=\sum c_i|e_i\rangle$, each basis vector $|e_i\rangle$ corresponds to a Voronoi cell V_i, interpreted as a phase circle with area $a_i=\mu(V_i)$. The system state can be expressed as $V=\sum c_iV_i$, where $c_i=\sqrt{a_i}e^{i\theta_i}$. By comparing the forms $|\psi\rangle=\sum c_i|e_i\rangle$ and $V=\sum c_iV_i$, we conclude that quantum probability $P(|e_i\rangle)=|c_i|^2=a_i$ is independent of the specific eigenstate $|e_i\rangle$, depending solely on the complex coefficient c_i. Thus, the phase circle V_i effectively replaces $|e_i\rangle$, with the probability of $|e_i\rangle$ equating to that of V_i.

Similarly, any classical probability distribution $A=\{A_1,A_2,\ldots,A_n\}$, where a_i is the probability of event A_i, can be expressed as a linear combination $A=\sum c_i A_i$, with $c_i=r_i e^{i\theta_i}$, $r_i=\sqrt{a_i}$, and $|c_i|^2=a_i$. Since the probabilities satisfy non-negativity and normalization ($\sum a_i=1$), each event A_i corresponds to a phase circle V_i with area a_i. The system state $A=\sum c_i A_i$ mirrors $V=\sum c_i V_i$, indicating that classical probability $P(A_i)=a_i=|c_i|^2$ is independent of the specific event A_i, relying only on the complex coefficient c_i. Hence, the phase circle V_i can replace A_i, with $P(A_i)=P(V_i)$.

This correspondence reveals that any probability—classical or quantum—can be expressed as a linear combination $A=\sum c_i A_i$, with a corresponding phase circle form $V=\sum c_i V_i$. The probability $P(A_i)=|c_i|^2$ is thus abstracted from the specific nature of A_i (whether an eigenstate or event), determined solely by the squared modulus of the complex coefficient. On the other hand, $V=\sum c_i V_i$ represents a probability distribution structure. Consequently, there is a one-to-one correspondence between the probability distribution structures represented by all $V=\sum c_i V_i$ and those represented by all $A=\sum c_i A_i$, based on the probability distribution structure itself. Consequently, the probability of any event or eigenstate is equivalent to the probability of its corresponding phase circle, unifying the mathematical structure of probability across both domains. This abstraction aligns with the framework's emphasis on the shape invariance of probability and supports the topological interpretation of phase circles as universal carriers of probabilistic information.

5.4 Conditional Probability in the Geometric-Topological Framework

Conditional probability extends our framework by restricting the probability measure to a subset of the base space or Hilbert space.

Classical Conditional Probability: Consider a subset $B\subset A$, with measure $\mu(B)>0$. For any event V_i, the conditional probability given B is:

$$P(V_i|B)=\frac{\mu(V_i\cap B)}{\mu(B)}.$$

If $V_i\subseteq B$, then $\mu(V_i\cap B)=a_i$, so $P(V_i|B)=\frac{a_i}{\mu(B)}$. More generally, if $V_i\cap B\neq\emptyset$, $\mu(V_i\cap B)$ reflects the overlap. Define new complex coefficients:

$$c_{i'}=\sqrt{\frac{\mu(V_i\cap B)}{\mu(B)}}e^{i\theta_i},\quad P(V_i|B)=|c_{i'}|^2.$$

Example: Let A be divided into four Voronoi cells V_1,V_2,V_3,V_4, with $a_1=0.2,a_2=0.3,a_3=0.3,a_4=0.2$. Let $B=V_1\cup V_2$, so $\mu(B)=0.5$. For event $A=V_2\cup V_3$, since $A\cap B=V_2$, $\mu(A\cap B)=0.3$, thus:

$$P(A|B)=\frac{0.3}{0.5}=0.6.$$

For $V_2\subseteq B$, $P(V_2|B)=\frac{0.3}{0.5}=0.6$, with $c_{2'}=\sqrt{0.6}e^{i\theta_2}$.

Quantum Conditional Probability: In Hilbert space $\mathcal{H}$, event B corresponds to projection onto subspace $\mathcal{H}_B=\text{span}\{|e_j\rangle\}_{j\in J}$. For state $|\psi\rangle=\sum c_i|e_i\rangle$, the conditional probability is:

$$P(|e_i\rangle|B)=\frac{|c_i|^2}{\sum_{j\in J}|c_j|^2}=\frac{a_i}{\sum_{j\in J}a_j},\quad i\in J.$$

The new coefficients are $c_{i'}=\frac{c_i}{\sqrt{\sum_{j\in J}a_j}}$. For a spin-1/2 system with $|\psi\rangle=\sqrt{0.7}e^{i\theta_1}|\uparrow\rangle+\sqrt{0.3}e^{i\theta_2}|\downarrow\rangle$, and $B=|\uparrow\rangle\langle\uparrow|$, we have $P(|\uparrow\rangle|B)=1$, $c_{1'}=e^{i\theta_1}$.

Continuous Case: For a tubular structure T, conditional probability restricts the measure to a subregion, e.g., $s\in[s_1,s_2]$. The conditional probability density is:

$$P(s|B)=\frac{a(s)}{\int_{s_1}^{s_2}a(s')\,ds'},\quad c'(s)=\sqrt{\frac{a(s)}{\int_{s_1}^{s_2}a(s')\,ds'}}e^{i\theta(s)}.$$

Example: For a uniform tubular structure with $a(s)=1$ over $s\in[0,1]$, and $B=[0.2,0.8]$, the conditional density is:

$$P(s|B)=\frac{1}{\int_{0.2}^{0.8}1\,ds'}=\frac{1}{0.6},\quad s\in[0.2,0.8].$$

Thus, $c'(s)=\sqrt{\frac{1}{0.6}}e^{i\theta(s)}\approx1.291e^{i\theta(s)}$, and $P(s|B)=|c'(s)|^2=\frac{1}{0.6}$.

5.5 Bayesian Inference in the Geometric-Topological Framework

Bayesian inference extends our framework by updating probabilities based on observations, expressed through geometric rescaling and complex coefficients.

Classical Bayesian Inference: Consider hypotheses H_i corresponding to Voronoi cells V_i, with prior probabilities $P(H_i)=a_i=|c_i|^2$. For an observation D, the likelihood $P(D|H_i)$ can be defined based on the specific model. For instance, if D is a subset of A, $P(D|H_i)=\frac{\mu(V_i\cap D)}{a_i}$. The posterior probability is:

$$P(H_i|D)=\frac{P(D|H_i)P(H_i)}{P(D)}=\frac{\mu(V_i\cap D)}{\mu(D)}=a_{i'},$$

with $P(D)=\mu(D)$, and new coefficients $c_{i'}=\sqrt{a_{i'}}e^{i\theta_i}$. Example: With V_1,V_2,V_3, $a_1=0.4,a_2=0.3,a_3=0.3$, let D have $\mu(D)=0.5$, $\mu(V_1\cap D)=0.2$, $\mu(V_2\cap D)=0.2$, $\mu(V_3\cap D)=0.1$. Then:

$$P(H_1|D)=\frac{0.2}{0.5}=0.4,\quad P(H_2|D)=0.4,\quad P(H_3|D)=0.2.$$

Quantum Bayesian Inference: For state $|\psi\rangle=\sum c_i|e_i\rangle$, observation D is a projection onto subspace $\mathcal{H}_D=\text{span}\{|e_j\rangle\}_{j\in J}$. The posterior probability is:

$$P(H_i|D)=\frac{a_i}{\sum_{j\in J} a_j},\quad i\in J,$$

with coefficients $c_{i}'=\sqrt{\frac{a_i}{\sum_{j\in J} a_j}}e^{i\theta_i}$.

Continuous Case: In a tubular structure T, the prior density is $P(s)=a(s)$. Given observation D, the likelihood $P(D|s)=f(s,D)$, and the posterior density is:

$$P(s|D)=\frac{f(s,D)a(s)}{\int_0^1 f(s',D)a(s')\,ds'},$$

with coefficients $c'(s)=\sqrt{\frac{f(s,D)a(s)}{\int_0^1 f(s',D)a(s')\,ds'}}e^{i\theta(s)}$. Example: For a uniform tubular structure with $a(s)=1$, suppose D is observed at $s=0.5$, with likelihood $f(s,D)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(s-0.5)^2}{2\sigma^2}}$. The posterior is:

$$P(s|D)=\frac{e^{-\frac{(s-0.5)^2}{2\sigma^2}}}{\int_0^1 e^{-\frac{(s'-0.5)^2}{2\sigma^2}}\,ds'},$$

which approximates a Gaussian centered at $s=0.5$, truncated to $[0,1]$.

6 Conclusion

This paper constructs a probability theory framework from the principle of equal likelihood, using Voronoi diagrams and topology to reveal the abstract essence of probability. Key contributions include:

Defining V_i's probability as the area ratio a_i, introducing complex coefficients $c_i=\sqrt{a_i}e^{i\theta_i}$ via phase circle equivalence, such that $P(V_i)=|c_i|^2$.

Extending to continuous distributions using three-dimensional tubular structures, where cross-sectional phase circles represent probability density.

Incorporating conditional probability and Bayesian inference, enabling dynamic updates within the geometric-topological structure.

Providing a geometric-topological interpretation for the Born rule [3], unifying classical and quantum probability through linear combinations and phase circle correspondence.

The incorporation of continuous distributions, conditional probability, and Bayesian inference validates the framework's universality, potentially positioning it as a versatile alternative to traditional probability theory in geometric and quantum contexts. Future research may explore topological properties of higher-dimensional Voronoi diagrams, computational optimizations for tubular structures, and applications in quantum state reconstruction, complex system analysis, and cross-disciplinary probability modeling.

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